CHAPTER 10. A QUICK REVIEW OF DISTRIBUTIONS RELEVANT IN FINANCE

Contents

In this Appendix, we quickly review the properties of distributions relevant in finance, like

- Normal distribution,
- Lognormal distribution,
- Chi-Square distribution,
- Non-Central Chi-Square distribution,
- Poisson distribution,
- Exponential distribution,
- Gamma distribution,
- Multivariate Gaussian distribution.

Then we present three important numerical procedures

- Simulation of random variables, via the inverse method.
- numerical integration;
- numerical inversion of the characteristic function.

10.1. The Normal Distribution

Fact 74 (Normal Distribution) *A normal (Gaussian) random variable on $\Re$ with expected value $\mu \in \Re$ and standard deviation $\sigma \in \Re^+$ has density function (PDF) $\phi_{\mu,\sigma}(x)$ and cumulative distribution (CDF) $\Phi_{\mu,\sigma}(x)$ given by:*

$$\phi_{\mu,\sigma}(x) = f_X(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right), x \in \Re,$$

and

$$\Phi_{\mu,\sigma}(x) = F_X(x;\mu,\sigma) = \int_{-\infty}^{x} \phi_{\mu,\sigma}(s)\, ds.$$

We write $X \sim \mathcal{N}(\mu,\sigma^2)$. If $\mu = 0$ and $\sigma = 1$ we have the so called unit standard Gaussian random variable. In particular, if $Z \sim \mathcal{N}(0,1)$, then

$$X = \mu + \sigma Z \sim \mathcal{N}(\mu,\sigma^2).$$

Given the CDF, we can compute the probability that X falls in a given interval

$$\Pr(a < X < b) = \Phi_{0,1}\left(\frac{b-\mu}{\sigma}\right) - \Phi_{0,1}\left(\frac{a-\mu}{\sigma}\right).$$

We can produce Table (10.1). Table (10.2) illustrates the main properties of the Gaussian distribution. The resulting PDF and CDF are shown in Figure (10.1).

10.2. The Lognormal distribution

Fact 75 (Lognormal Distribution) *Let X be a Normal r.v. with mean μ and standard deviation σ. Then the r.v.*

$$Y = \exp(X),$$

is said to have a `lognormal distribution` *with parameters μ and σ and density $f_Y(y;\mu,\sigma)$ given by*

$$f_Y(y;\mu,\sigma) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{\ln y-\mu}{\sigma}\right)^2\right), y \in \Re^+.$$

Probability Intervals			
k	$\mu - k\sigma$	$\mu + k\sigma$	$\Pr(\mu - k\sigma < X < \mu + k\sigma)$
1	-0.3542	0.2742	0.6827
2	-0.6684	0.5884	0.9545
3	-0.9826	0.9026	0.9973
4	-1.2968	1.2168	0.9999
5	-1.611	1.531	1.0000

Table 10.1: Probability intervals of the Gaussian distribution.

Properties of the Normal distribution	
Support	$x \in (-\infty; +\infty)$
Mean	$\mathbb{E}(X) = \mu$
Variance	$\mathbb{V}ar(X) = \sigma^2$
Skewness	$\frac{\mathbb{E}\left((X-\mu)^3\right)}{\sigma^3} = 0$
Kurtosis	$\frac{\mathbb{E}\left((X-\mu)^4\right)}{\sigma^4} = 3$
Central odd moments	$\mathbb{E}\left((X-\mu)^{2n+1}\right) = 0$
Central even moments	$\mathbb{E}\left((X-\mu)^{2n}\right) = \frac{(2n)!}{n!}\frac{\sigma^{2n}}{2^n}$

Table 10.2: Moments of the Gaussian distribution.

Moreover:

$$\mathbb{E}(Y) = e^{\mu + \frac{1}{2}\sigma^2}$$

and

$$\mathbb{V}ar(Y) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right).$$

We write $Y \sim \mathcal{LN}(\mu, \sigma^2)$.
The relationship between densities of the Gaussian and Lognormal distribution is depicted in Figure (10.2).

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%PLOT OF THE GAUSSIAN DISTRIBUTION%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%assign parameters
mu=0.2;
sg=0.1;
%assign x-range
x=linspace(-.4,0.8,100);
%compute pdf and cdf
pdfg=pdf('norm',x,mu,sg);
cdfg=cdf('norm',x,mu,sg);
%make the plot
plot(x,pdfg,'r',x,cdfg,'b');
%define the limits in the x-axis
xlim([-0.4 0.8])
%insert the legend
legend('Gaussian pdf','Gaussian cdf')
%print the figure
print(h,'-dpng','FigGaussianpdf')
```

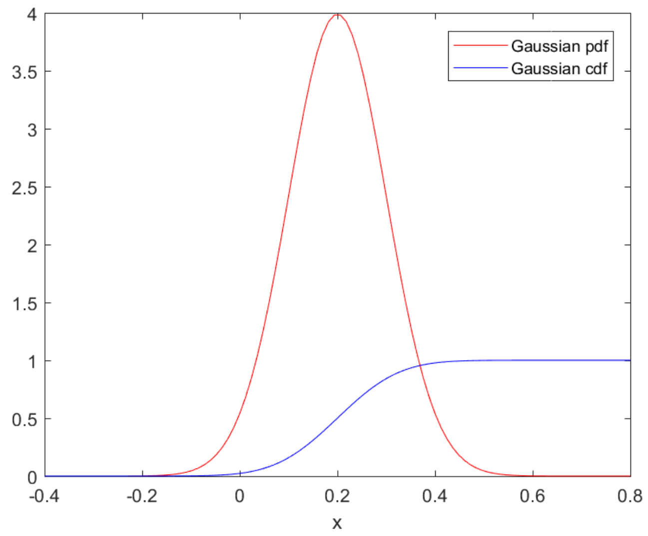

Figure 10.1: Density and Cumulative Distribution of the Gaussian random variable.

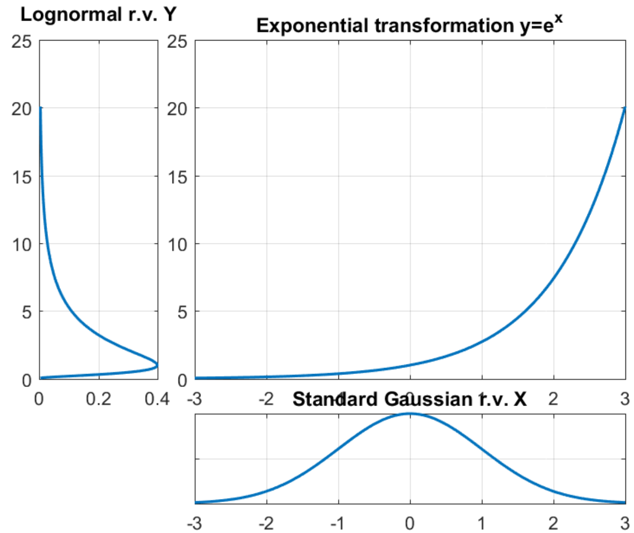

Figure 10.2: Densities of the Gaussian and of the Lognormal random variables.

The cumulative density function of a Lognormal r.v. is given by:

$$\begin{aligned}
F_Y(y) &= \Pr(Y \leq y) \\
&= \Pr\left(e^X \leq y\right), \textit{ where } X \sim \mathcal{N}\left(\mu, \sigma^2\right) \\
&= \Pr(X \leq \ln(y)) \\
&= \Phi_{\mu,\sigma^2}(\ln(y)) \\
&= \Phi_{0,1}\left(\frac{\ln(y) - \mu}{\sigma}\right).
\end{aligned}$$

Given the CDF, we can compute the probability that X falls in a given interval

$$\begin{aligned}
\Pr(a < Y < b) &= \Phi_{\mu,\sigma^2}(\ln(b)) - \Phi_{\mu,\sigma^2}(\ln(a)) \\
&= \Phi_{0,1}\left(\frac{\ln(b) - \mu}{\sigma}\right) - \Phi_{0,1}\left(\frac{\ln(a) - \mu}{\sigma}\right).
\end{aligned}$$

The main properties of the Lognormal distribution are summarized in Table (10.3). The resulting PDF and CDF are shown in Figure (10.3).

LOGNORMAL DISTRIBUTION $Y \sim \mathcal{LN}(\mu, \sigma^2)$ FACTS	
Support	
	$y \in [0; +\infty)$
PDF	
	$f_Y(y; \mu, \sigma) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2\right), y \in \Re^+.$
CDF	
	$\Phi_{0,1}\left(\frac{\ln(y) - \mu}{\sigma}\right)$
Mean	
	$\mathbb{E}(Y) = e^{\mu + \frac{1}{2}\sigma^2}$
Variance	
	$\mathbb{V}ar(Y) = e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right)$
Skewness	
	$\frac{\mathbb{E}\left((Y - \mathbb{E}(Y))^3\right)}{(\mathbb{V}ar(Y))^{\frac{3}{2}}} = \left(e^{\sigma^2} + 2\right)\sqrt{e^{\sigma^2} - 1}$
Kurtosis	
	$\frac{\mathbb{E}\left((Y - \mathbb{E}(Y))^4\right)}{(\mathbb{V}ar(Y))^2} = e^{4\sigma^2} + 2e^{3\sigma^2} + 3e^{2\sigma^2} - 3$
Moments	
	$\mathbb{E}(Y^n) = e^{n\mu + \frac{1}{2}n^2\sigma^2}$

Table 10.3: Properties of the Lognormal distribution.

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%PLOT OF THE LOGNORMAL DISTRIBUTION%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%assign parameters
mu=0.2;  sg=0.1;
%assign x-range
x=linspace(0.0,2,100);
%compute pdf and cdf
pdflg=pdf('lognorm',x,mu,sg);
cdflg=cdf('lognorm',x,mu,sg);
%make the plot
h=figure('Color', [ 1 1 1])
plot(x,pdflg,'r',x,cdflg,'b');
%define the limits in the x-axis
xlim([0 2]); xlabel('x');
%insert title
title('Lognormal pdf and cdf when \mu=0.2 and \sigma =0.1')
%insert the legend
legend('Log-normal pdf','Log-normal cdf')
%print the figure
print(h,'-dpng','FigLogGaussianpdf.png')
```

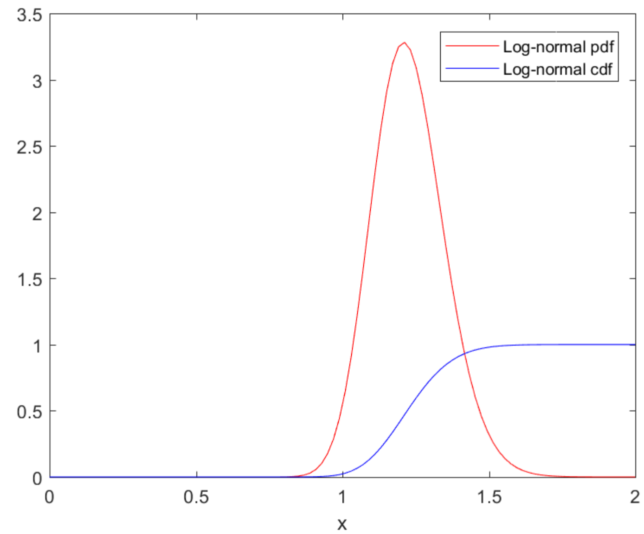

Figure 10.3: Density and Cumulative Distribution of the Lognormal random variable.

10.3. THE CHI-SQUARE DISTRIBUTION

Fact 76 (Chi-Square Distribution) *Let X_i, i=1,...,n, be n independent Normal random variables with zero mean and unit standard deviation. Then the r.v.*

$$Y = \sum_{i=1}^{n} X_i^2,$$

is said to have a `chi-square distribution` *with n degrees of freedom and density $f_Y(y;n)$ given by*

$$f_Y(y;n) = \frac{e^{-y/2} y^{n/2-1}}{2^{n/2}\Gamma(n/2)}, y \in \Re^+.$$

where $\Gamma(n)$ is the Gamma function

$$\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} dt,$$

and if n is an integer $\Gamma(n) = (n-1)!$. Moreover:

$$\mathbb{E}(X) = n$$

and

$$\mathbb{V}ar(X) = 2n.$$

We write $Y \sim \chi_n^2$.

The main properties of the Chi-square distribution are summarized in Table (10.4). The resulting PDF and CDF are shown in Figure (10.4). Figure (10.5) illustrates the different shapes of the chi-square distribution for increasing number of degrees of freedom.

Properties of the chi-square distribution	
Support	$y \in [0; +\infty)$
Mean	$\mathbb{E}(Y) = n$
Variance	$\mathbb{V}ar(Y) = 2n$
Skewness	$\frac{\mathbb{E}\left((Y-n)^3\right)}{(2n)^{\frac{3}{2}}} = \sqrt{\frac{8}{n}}$
Excess Kurtosis	$\frac{\mathbb{E}\left((Y-n)^4\right)}{(2n)^2} - 3 = \frac{12}{n}$
Moments	$\mathbb{E}(Y^m) = 2^m \frac{\Gamma\left(m+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$

Table 10.4: Moments of the Chi-square distribution.

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%THE Chi-2 DISTRIBUTION AND THE DEGREES OF FREEDOM%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all;
%assign parameters
n=[5 10 20 50]; %degrees of freedom
%assign x-range
x=linspace(0,100,300);
%compute pdf and cdf
for i=1:length(n)
    pdfchi2(i,:)=pdf('chi2',x,n,n(i));
end
h=figure('Color', [ 1 1 1])
plot(x,pdfchi2,'.')
xlabel('x')
legend('n=5','n=10','n=20','n=50')
title('The Chi-Square pdf varying ...
the number n of degrees of freedom')
```

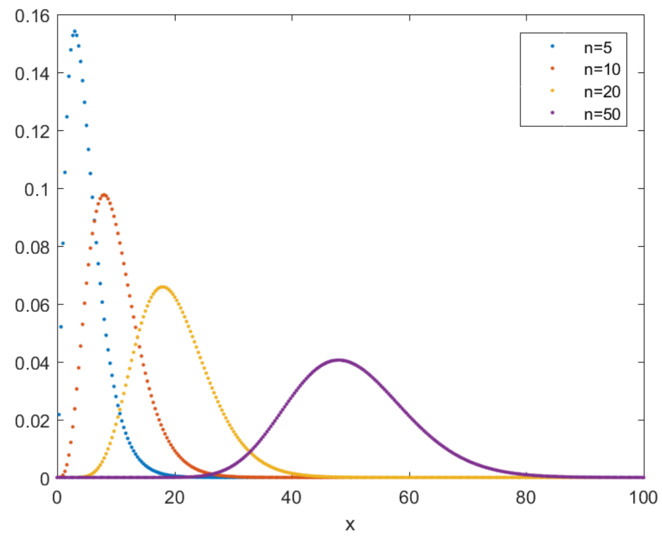

Figure 10.5: Density of the Chi-Square random variable varying n.

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%PLOT OF THE Chi−2 DISTRIBUTION%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all;
%assign parameters
n=5; %degrees of freedom
%assign x−range
x=linspace(0,20,100);
%compute pdf and cdf
pdfchi2=pdf('chi2',x,n);
cdfchi2=cdf('chi2',x,n);
%make the plot
h=figure('Color', [ 1 1 1])
plot(x,pdfchi2,'r',x,cdfchi2,'b')
xlabel('x')
title('Chi−Square PDF and CDF')
legend('Chi−Square pdf','Chi−Square cdf')
print(h,'−djpeg','FigChiSquarepdf.jpg')
```

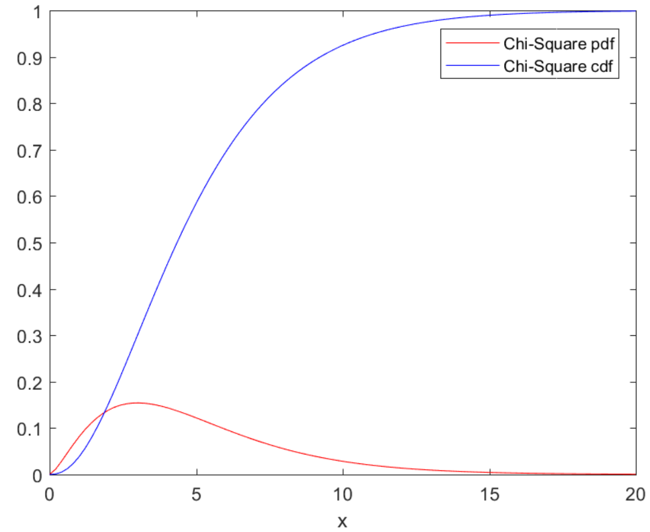

Figure 10.4: Density and Cumulative Distribution of the Chi-Square random variable with $n = 20$ degrees of freedom.

10.4. THE NONCENTRAL CHI-SQUARED DISTRIBUTION

Fact 77 (Noncentral chi-squared distribution) *Let X_i, $i = 1, \ldots, n$, be n independent Gaussian random variables with zero mean and unit standard deviation. Then the r.v.*

$$Y = \sum_{i=1}^{n} (X_i + \delta_i)^2,$$

is said to have a `noncentral chi-squared distribution` *with n degrees of freedom and non-centrality parameter d*

$$d = \sum_{i}^{n} \delta_i^2,$$

and density $f_Y(y;n,d)$ given by

$$f_Y(y;n,d) = e^{-(y+d)/2}\frac{x^{n/2-1}}{2^{n/2}}\sum_{k=0}^{\infty}\frac{d^k}{2^{2k}k!\Gamma(k+1/2n)}, y \in \Re^+.$$

An equivalent representation of the density function is in terms of the Bessel function

$$f_Y(y;n,d) = \frac{e^{-(y+d)/2}y^{(n-1)/2}\sqrt{d}}{2(dy)^{n/4}}I_{n/2-1}\left(\sqrt{dy}\right),$$

where $I_n(x)$ is a modified Bessel function of the first kind.

Moreover:

$$\mathbb{E}(Y) = n+d,$$

and

$$\mathbb{V}ar(Y) = 2(n+2d).$$

We write $Y \sim \chi^2_{n,d}$.

Remark 78 *We can make the following observations:*

- *The noncentrality parameter depends on the sum $d = \sum_i^n \delta_i^2$, not on the individual values of the parameters δ_i;*
- *If $d = 0$, the noncentral chi-squared distribution collapses to the chi-squared distribution.*
- *Here we are assuming that n is an integer parameter: this is not necessary.*

Table (10.5)illustrates the main properties of the noncentral chi-squared distribution. The resulting PDF and CDF are shown in Figure (10.6). Figure (10.7) illustrates the different shapes of the density for different values of the of noncentrality parameter d.

Properties of the noncentral chi-squared distribution distribution	
Support	$y \in [0; +\infty)$
Mean	$\mathbb{E}(Y) = n + d$
Variance	$\mathbb{V}ar(Y) = 2 \times (n + 2d)$
Skewness	$\frac{\mathbb{E}\left((Y-\mathbb{E}(Y))^3\right)}{(\mathbb{V}ar(Y))^{\frac{3}{2}}} = \frac{2^{3/2}(n+3d)}{(n+2d)^{3/2}}$
Excess Kurtosis	$\frac{\mathbb{E}\left((Y-\mathbb{E}(Y))^4\right)}{(\mathbb{V}ar(Y))^2} - 3 = \frac{12(n+4d)}{(n+2d)^2}$

Table 10.5: Moments of the noncentral chi-squared distribution.

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%PLOT OF THE NONCENTRAL CHI-SQUARED DISTRIBUTION%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all;
%assign parameters
n=5; %degrees of freedom
noncentralpar=2;%non centrality parameter
%assign x-range
x=linspace(0,20,100);
%compute pdf and cdf
pdfncchi2=pdf('ncx2',x,n,noncentralpar );
cdfncchi2=cdf('ncx2',x,n,noncentralpar);
%make the plot
h=figure('Color', [ 1 1 1])
plot(x,pdfncchi2,'r',x,cdfncchi2,'b')
xlabel('x')
title('Non central Chi-Square pdf and cdf when n=5 and d =2')
legend('Non Central Chi-Square pdf',...
'Non Central Chi-Square cdf')
print(h,'-djpeg','FigNCChiSquarepdf.jpg')
```

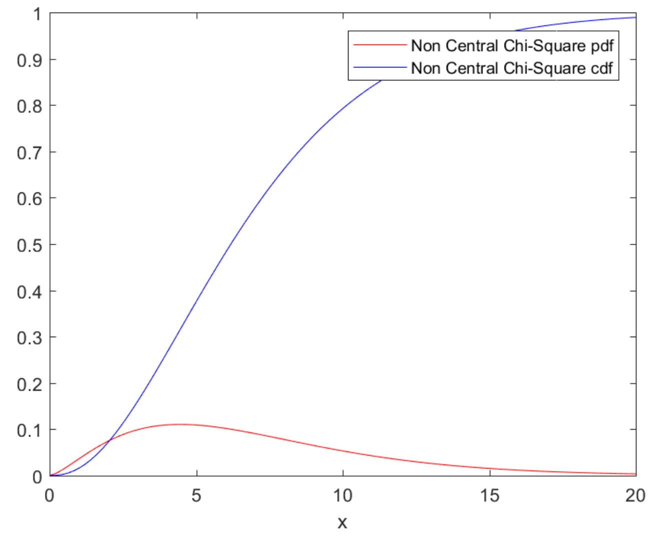

Figure 10.6: Density and CDF of the noncentral chi-squared distribution with $n = 5$ and $d = 2$.

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%THE NCChi-2 DISTRIBUTION AND THE %%%%
%%%%  PARAMETER OF NON CENTRALITY    %%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all;
%assign parameters
n=15; %degrees of freedom
ncp=[0.5  1.5 3 5 10];%non centrality parameter
%assign x-range
x=linspace(0,20,300);
%compute pdf and cdf
for i=1:length(ncp)
    pdfncchi2(i,:)=pdf('ncx2',x,ncp(i));
end
h=figure('Color', [ 1 1 1])
plot(x,pdfncchi2,'.')
xlabel('x')
legend('d=0.5','d=1.5','d=3','d=5','d=10')
title('The Non-Central Chi-Square pdf varying the non ...
centrality parameter and with n=15 of degrees of freedom')
print(h,'-djpeg','FigNCChiParSquarepdf.jpg')
```

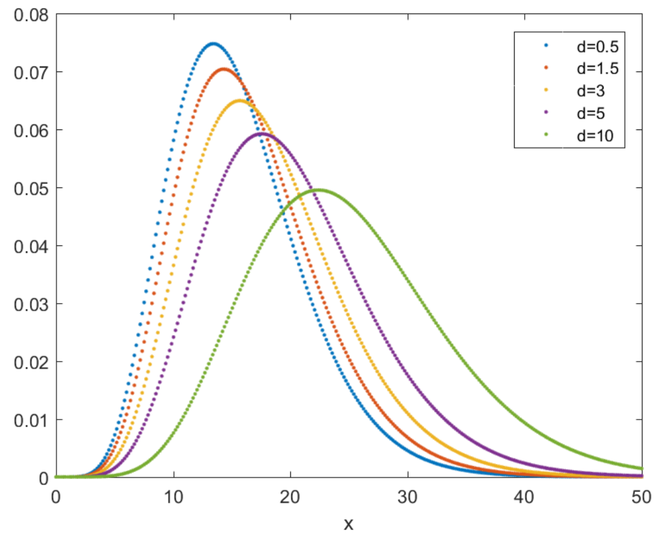

Figure 10.7: Density of the non-central chi-square distribution varying the parameter of non centrality d.

Properties of the Poisson distribution distribution	
Support	$n \in 0, 1, 2, \cdots$
Mean	$\mathbb{E}(N) = \lambda$
Variance	$\mathbb{V}ar(N) = \lambda$
Skewness	$\frac{\mathbb{E}\left((N-\lambda)^3\right)}{\lambda^{\frac{3}{2}}} = \frac{1}{\sqrt{\lambda}}$
Excess Kurtosis	$\frac{\mathbb{E}\left((N-\lambda)^4\right)}{\lambda^2} - 3 = \frac{1}{\lambda}$

Table 10.6: Moments of the Poisson distribution.

10.5. THE POISSON DISTRIBUTION

Fact 79 (Poisson Distribution) *A Poisson random variable N with rate λ has probability mass*

$$p_N(n) = \frac{e^{-\lambda}\lambda^n}{n!}.$$

Moreover, the mean and the variance are respectively:

$$\mathbb{E}(N) = \lambda$$

and

$$\mathbb{V}ar(N) = \lambda.$$

We write $N \sim Poi(\lambda)$.

Table (10.6)illustrates the main properties of the Poisson distribution. The resulting probability mass function and CDF are shown in Figure (10.8). The impact of the rate of arrival λ is represented in Figure (10.9).

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%PLOT OF THE Poisson DISTRIBUTION%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all;
%assign parameters
lambda=5; %rate of arrival
%assign x-range
x=(0:1:20);
%compute pdf and cdf
pdfpoil=poisspdf(x,lambda);
cdfpoil=poisscdf(x,lambda);
%make the plot
h=figure('Color', [ 1 1 1])
plot(x,pdfpoil,'--o r',x,cdfpoil,'--o b')
xlabel('x')
title('Poisson pdf and cdf when \lambda = 5')
legend('Poisson pdf',...
'Poisson cdf')
print(h,'-djpeg','FigPoisson1pdf.jpg')
```

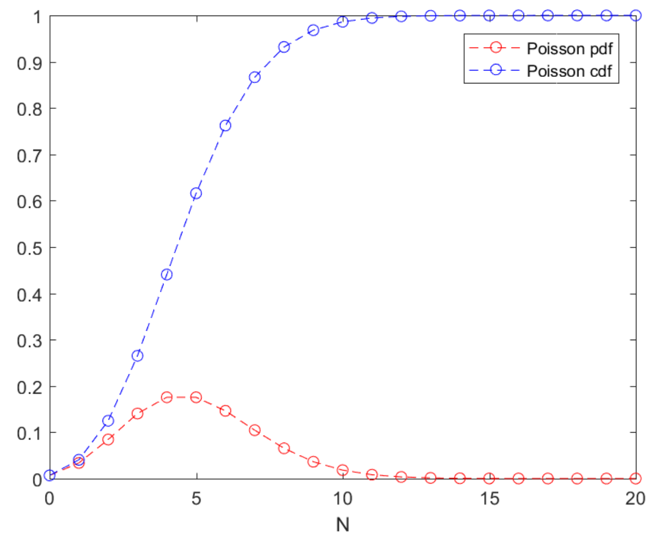

Figure 10.8: Density and CDF of the Poisson distribution with $\lambda = 5$. The functions are only defined at integer values of x. The connecting lines are only guides for the eye.

10.6. The Exponential distribution

Fact 80 (Exponential Distribution) *A non negative random variable X is said to have an* `exponential distribution` *with parameter λ when its density is given by*

$$f_X(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0 & x < 0, \end{cases}$$

and it CDF is given by

$$F_X(x;\lambda) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0, \\ 0 & x < 0, \end{cases}$$

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%The  Poisson DISTRIBUTION and the rate of arrival%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all;
%assign parameters
lambda=[0.5 1 5 10]; %rate of arrival
%assign x-range
x=(0:1:20);
%compute pdf and cdf
for i=1:length(lambda)
    pdfpoil(i,:)=poisspdf(x,lambda(i));
    cdfpoil(i,:)=poisscdf(x,lambda(i));
end
%make the plot
h=figure('Color', [ 1 1 1])
subplot(2,1,1)
plot(x,pdfpoil,'--o')
xlabel('x')
title('Poisson pdf varying the rate of arrival \lambda')
legend('\lambda = 0.5','\lambda = 1',...
'\lambda = 5','\lambda = 10')
subplot(2,1,2)
plot(x,cdfpoil,'o')
xlabel('x')
title('Poisson cdf varying the rate of arrival \lambda')
print(h,'-djpeg','FigPoisson2pdf.jpg')
```

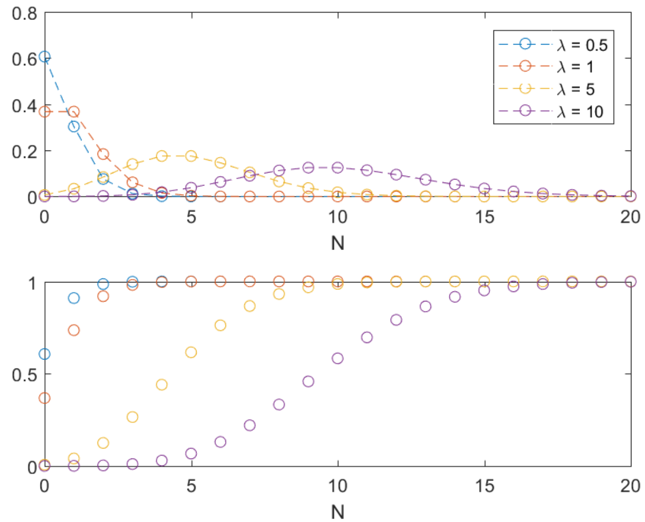

Figure 10.9: PDF (top) and CDF (bottom) of the Poisson distribution varying the rate of arrival λ.

Moreover:

$$\mathbb{E}(X) = \lambda^{-1}$$

Properties of the Exponential distribution distribution	
Support	$x \in [0; +\infty)$
Mean	$\mathbb{E}(X) = \frac{1}{\lambda}$
Variance	$\mathbb{V}ar(X) = \frac{1}{\lambda^2}$
Skewness	$\frac{\mathbb{E}\left((X-\mathbb{E}(X))^3\right)}{(\mathbb{V}ar(X))^{\frac{3}{2}}} = 2$
Kurtosis	$\frac{\mathbb{E}\left((X-\mathbb{E}(X))^4\right)}{(\mathbb{V}ar(X))^2} = 9$

Table 10.7: Moments of the Exponential distribution.

and

$$\mathbb{V}ar(X) = \lambda^{-2}.$$

We write $X \sim Exp(\lambda)$.

Table (10.7) illustrates the main properties of the Exponential distribution, whilst PDF and CDF are shown in Figure (10.10).

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%PLOT OF THE Exponential DISTRIBUTION%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all;
%assign parameters
lambda=1.5; %rate of arrival
%assign x-range
x=linspace(0,5,100);
%compute pdf and cdf
pdfexpl=exppdf(x,lambda^(-1));
cdfexpl=expcdf(x,lambda^(-1));
%make the plot
h=figure('Color', [ 1 1 1])
plot(x,pdfexpl,'- r',x,cdfexpl,'- b')
xlabel('x')
title('Exponential pdf and cdf when \lambda = 1.5')
legend('Exponential pdf',...
'Exponential cdf')
print(h,'-djpeg','Figexpopdf.jpg')
```

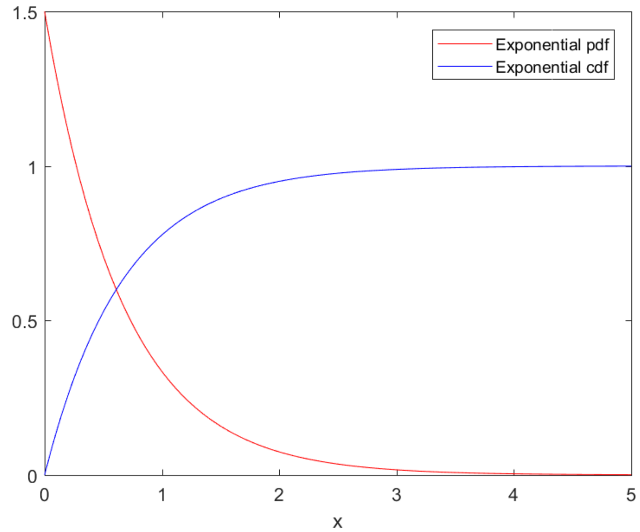

Figure 10.10: Density and CDF of the Exponential distribution with $\lambda = 1.5$.

10.7. The Gamma distribution

Fact 81 (Gamma Distribution) *A non negative random variable X is said to have a* `Gamma distribution` *with shape parameter α and rate parameter λ when its density is given by*

$$f_X(x;\alpha,\lambda) = \frac{1}{\Gamma(\alpha)}\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x},$$

where $\Gamma(\alpha)$ is the Gamma function, which is defined as

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}e^{-x}dx,$$

Moreover:

$$\mathbb{E}(X) = \frac{\alpha}{\lambda}$$

and

$$\mathbb{V}ar(X) = \frac{\alpha}{\lambda^2}$$

We write $X \sim \Gamma(\alpha,\lambda)$.

Remark 82
- *A Gamma random variable with parameter $\lambda = \frac{1}{2}$, i.e. $X \sim \Gamma(\alpha, \frac{1}{2})$, is distributed according to a chi-square distribution with 2α degrees of freedom;*
- *A Gamma random variable with $\alpha = 1$, i.e. $X \sim \Gamma(1,\lambda)$, is distributed according to an exponential distribution of parameter λ.*

Table (10.8)illustrates the main properties of the Gamma distribution. Figures (10.25) and (10.12) show the Gamma PDF and CDF for different values of the parameters α and λ.

Properties of the Gamma distribution distribution	
Support	$x \in [0;+\infty)$
Mean	$\mathbb{E}(X) = \frac{\alpha}{\lambda}$
Variance	$\mathbb{V}ar(X) = \frac{\alpha}{\lambda^2}$
Skewness	$\frac{\mathbb{E}\left((X-\mathbb{E}(X))^3\right)}{(\mathbb{V}ar(X))^{\frac{3}{2}}} = \frac{2}{\sqrt{\alpha}}$
Kurtosis	$\frac{\mathbb{E}\left((X-\mathbb{E}(X))^4\right)}{(\mathbb{V}ar(X))^2} = 3 + \frac{6}{\alpha}$

Table 10.8: Moments of the Gamma distribution.

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%PLOT OF THE Gamma DISTRIBUTION%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all;
%assign parameters
alpha=5;     %shape parameter
lambda=0.5; %rate parameter
%assign x-range
x=linspace(0,30,200);
%compute pdf and cdf
pdfgamma=gampdf(x,alpha,lambda^(-1));
cdfgamma=gamcdf(x,alpha,lambda^(-1));
%make the plot
h=figure('Color', [ 1 1 1])
plot(x,pdfgamma,'- r',x,cdfgamma,'- b')
xlabel('x')
title('Gamma pdf and cdf when\alpha =5 \lambda = 0.5')
legend('Gamma pdf',...
'Gamma cdf')
print(h,'-djpeg','FigGamma1pdf.jpg')
```

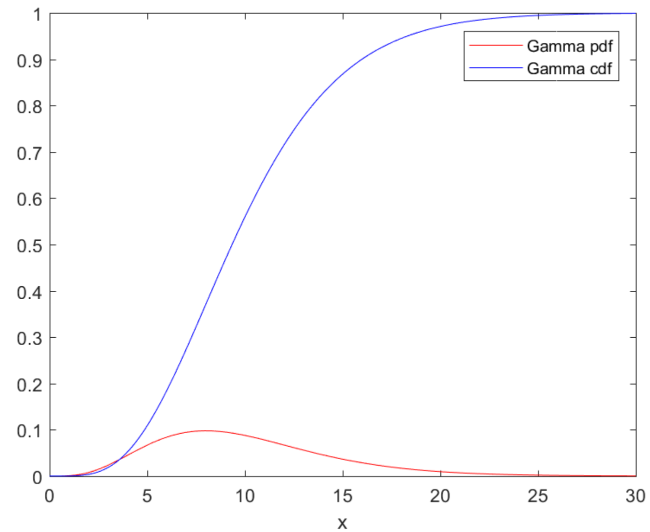

Figure 10.11: Density and CDF of the Gamma distribution with $\alpha = 5, \lambda = 0.5$.

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%The  Gamma DISTRIBUTION and the shape/rate parameters%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
alpha=[1 2 5 10];
lambda=[0.5 0.5 0.5 2]; %rate parameter
%assign x-range
x=linspace(0,30,200);
%compute pdf and cdf
for i=1:length(alpha)
    pdfgamma(i,:)=gampdf(x,alpha(i),lambda(i)^(-1));
    cdfgamma(i,:)=gamcdf(x,alpha(i),lambda(i)^(-1));
end
%make the plot
h=figure('Color', [ 1 1 1])
subplot(2,1,1)
plot(x,pdfgamma,'-')
xlabel('x')
title('Gamma pdf varying \alpha and \lambda')
legend('\alpha = 1, \lambda = 0.5','\alpha = 2, \lambda = 0.5',.
'\alpha = 5, \lambda = 0.5','\alpha = 10, \lambda = 2')
subplot(2,1,2)
plot(x,cdfgamma,'-')
xlabel('x')
title('Gamma cdf varying \alpha and \lambda')
legend('\alpha = 1, \lambda = 0.5','\alpha = 2, \lambda = 0.5',.
'\alpha = 5, \lambda = 0.5','\alpha = 10, \lambda = 2')
print(h,'-djpeg','FigGamma2pdf.jpg')
```

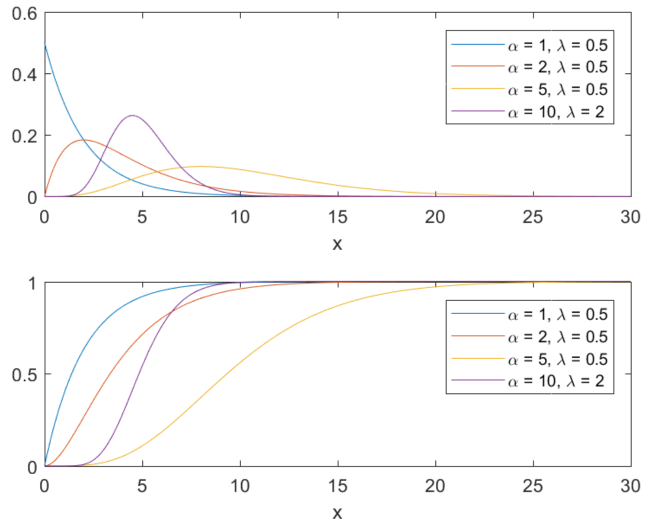

Figure 10.12: PDF (top) and CDF (bottom) of the Gamma distribution varying the shape parameter α and the rate parameter λ.

10.8. The multivariate Gaussian distribution

Fact 83 (Multivariate Gaussian Distribution) *Let $\boldsymbol{\mu}$ a $N \times 1$ vector and $\boldsymbol{\Sigma}$ a $N \times N$ matrix, symmetric and positive definite. A random vector $\mathbf{X} = (X_1, ..., X_N)$ is distributed according to a multivariate normal distribution with parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, and we write*

$$\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}),$$

if its density function $f_{\mathbf{X}}(\mathbf{x})$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^N \det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right),$$

where $\det(\boldsymbol{\Sigma})$ is the determinant of the matrix $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}^{-1}$ is the inverse of the matrix $\boldsymbol{\Sigma}$.

If $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then

$$\mathbb{E}(\mathbf{X}) = \boldsymbol{\mu}, \mathbb{V}(\mathbf{X}) = \boldsymbol{\Sigma},$$

and we call $\boldsymbol{\mu}$ the expected value (or mean) vector and $\boldsymbol{\Sigma}$ the covariance matrix. In particular this means that a multivariate Gaussian distribution is determined by its mean vector and covariance matrix.

The mean vector is a vector with N components. The ith component is denoted by μ_i and it represents the expected value of X_i

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_i \\ \vdots \\ \mu_N \end{bmatrix} = \begin{bmatrix} \mathbb{E}(X_1) \\ \\ \mathbb{E}(X_i) \\ \\ \mathbb{E}(X_N) \end{bmatrix}.$$

The covariance matrix $\boldsymbol{\Sigma}_{N \times N}$:

- *is a squared and symmetric matrix;*

- *it is positive definite, i.e.*
$$\mathbf{x}^{'}\boldsymbol{\Sigma}\mathbf{x} > 0\ \forall \mathbf{x} \in R^N, \mathbf{x} \neq \mathbf{0}.$$
- *it is made of* N *variances (on the diagonal and denoted by* σ_i^2*) and* $N \times (N-1)/2$ *covariances (denoted by* σ_{ij} *with* $\sigma_{ij} = \sigma_{ji}$*).*
$$\boldsymbol{\Sigma}_{N\times N} = \begin{bmatrix} \sigma_1^2 & & \cdots & & \sigma_{1,N} \\ & \sigma_2^2 & & & \\ \vdots & & \ddots & & \vdots \\ & & & \sigma_{N-1}^2 & \\ \sigma_{N,1} & & \cdots & & \sigma_N^2 \end{bmatrix}$$
where
$$\sigma_i^2 = \mathbb{V}(X_i)\,; \sigma_{ij} = \mathbb{C}ov\left(X_i, X_j\right) = \mathbb{E}\left((X_i - \mu_i)\left(X_j - \mu_j\right)\right).$$

Definition 84 *We say that the random vector* $\mathbf{Z} = (Z_1, ..., Z_N)$ *has a multivariate standard normal distribution if*

$$\boldsymbol{\mu} = \mathbf{0_N},\ \text{and}\ \boldsymbol{\Sigma} = \mathbf{I}_N,$$

where $\mathbf{0}_N$ *is a vector of zeros and* $\mathbf{I}_N$ *is the identity matrix of order* N*. It follows that* Z_i *and* Z_j *are uncorrelated and (being normal) also independent. The density function is simply*

$$f_Z(\mathbf{z}) = \frac{1}{\sqrt{(2\pi)^N}} \exp\left(-\frac{1}{2}\mathbf{z}'\mathbf{z}\right) = \frac{1}{\sqrt{(2\pi)^N}} \exp\left(-\frac{1}{2}\sum_{i=1}^{N} z_i^2\right).$$

10.8.1. The bivariate Normal distribution

If $N = 2$, the mean vector and the covariance matrix can be written as

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}; \boldsymbol{\Sigma}_{2\times 2} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}; \boldsymbol{\Sigma}_{2\times 2}^{-1} = \frac{1}{1-\rho^2}\begin{bmatrix} \frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1\sigma_2} \\ -\frac{\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix}$$

where $\rho = \mathbb{C}orr\left(X_1, X_2\right)$ and the density function becomes

$$f_X\left(\mathbf{x}\right) = \frac{\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)}{\sigma_1^2}^2 - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)}{\sigma_2^2}^2\right)\right)}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}.$$

The corresponding density is bell-shaped with maximum in (μ_1,μ_2).

The contour lines, i.e. the combinations of x_1 and x_2 such that the density is constant are described by the equation

$$-\frac{1}{2\left(1-\rho^2\right)}\left(\frac{\left(x_1-\mu_1\right)}{\sigma_1^2}^2 - 2\rho\frac{\left(x_1-\mu_1\right)\left(x_2-\mu_2\right)}{\sigma_1\sigma_2} + \frac{\left(x_2-\mu_2\right)}{\sigma_2^2}^2\right) = k,$$

that it is the equation of an ellipse. Figures (10.13) - (10.15) show the bivariate Normal density (left panel) and the corresponding contour plot (right panel) for different levels of the correlation, ρ.

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%PLOT OF THE BIVARIATE GAUSSIAN%%%%%%%%
%%%%%%%%POSITIVE CORRELATION%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%correlation
rho=0.9;
%variances and covariances
s1=0.3^2; s2=0.5^2; s12=rho*(s1*s2)^0.5;
factor=2*pi*(s1*s2)^0.5;
m=zeros(2,1); %mean vector
npoints1=50; npoints2=50;
x1=linspace(m(1)-3*s1^0.5,m(1)+3*s1^0.5,npoints1);
x2=linspace(m(2)-3*s2^0.5,m(2)+3*s2^0.5,npoints2);
[x1mesh, x2mesh]=...
meshgrid((x1-m(1))/s1^0.5,(x2-m(2))/s2^0.5);
x1x2mesh=-2*rho*x1mesh.*x2mesh;
pdfbiv=exp(-(x1mesh.^2+x1x2mesh+x2mesh.^2)...
/(2*(1-rho^2)))/(factor*(1-rho^2)^0.5);
figure1=figure('PaperSize',[20.98 29.68],'Color',[1 1 1]);
subplot(1,2,1); surf(x1mesh,x2mesh,pdfbiv)
axis square
title('Bivariate Normal pdf, \rho=0.9')
ylabel('x_1'),xlabel('x_2'),zlabel('PDF')
subplot(1,2,2); contour(x1mesh,x2mesh,pdfbiv)
xlabel('x_1'),ylabel('x_2')
title('Contour Level, \rho=0.9')
axis square
grid on
```

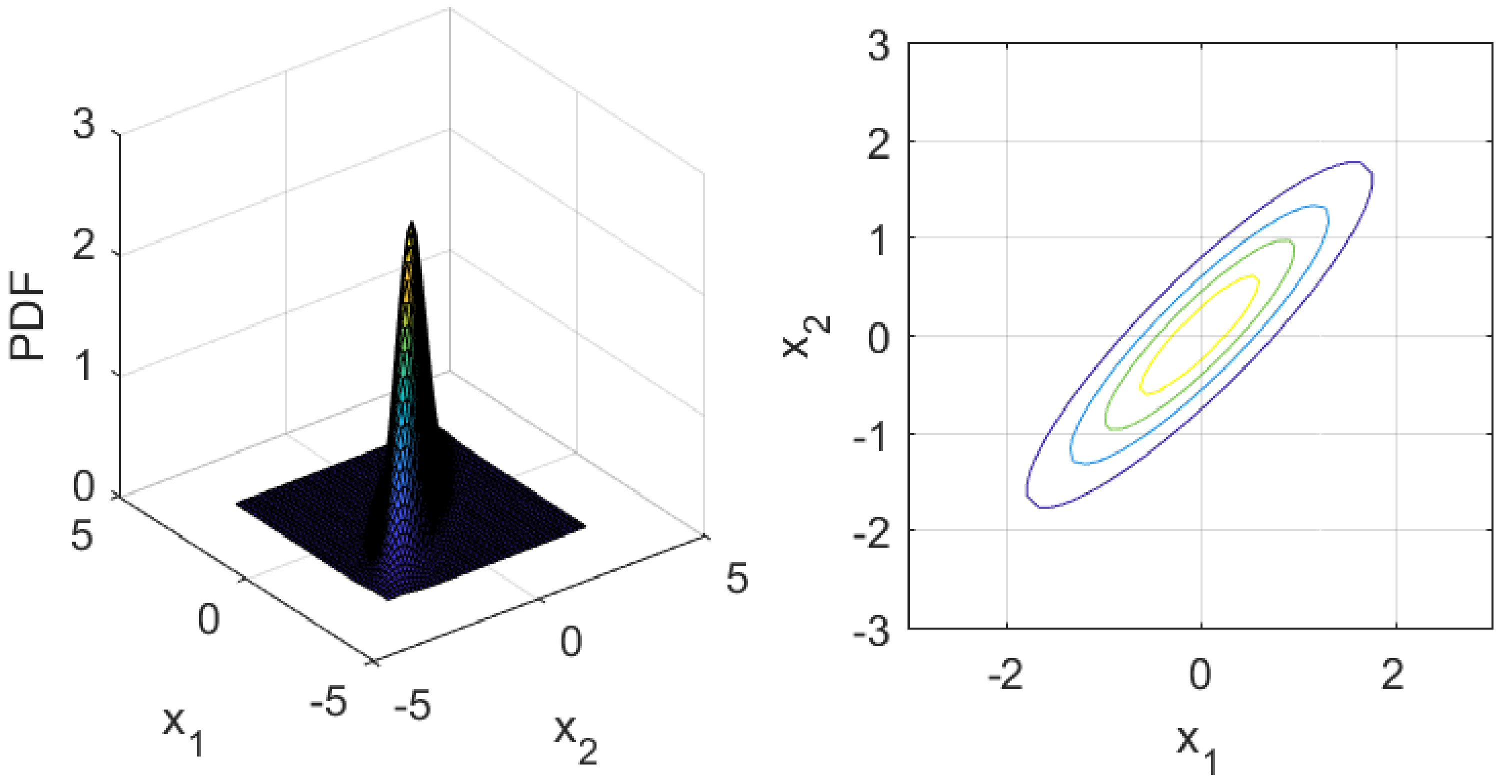

Figure 10.13: Left: Bivariate standard Gaussian distribution. Right: Contour Level. Parameter $\rho = 0.9$.

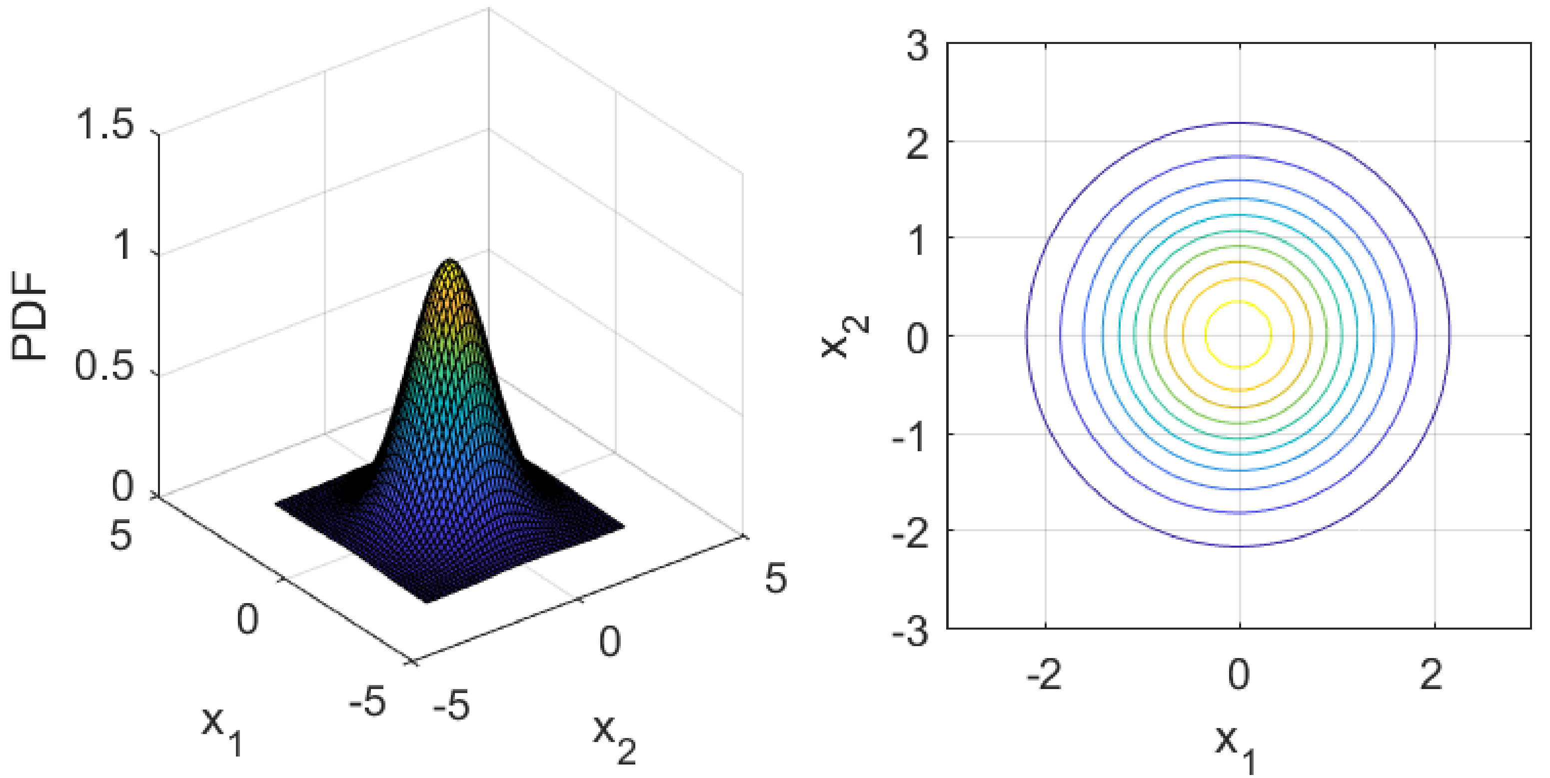

Figure 10.14: Left: Bivariate standard Gaussian distribution. Right: Contour Level. Parameter $\rho = 0.0$.

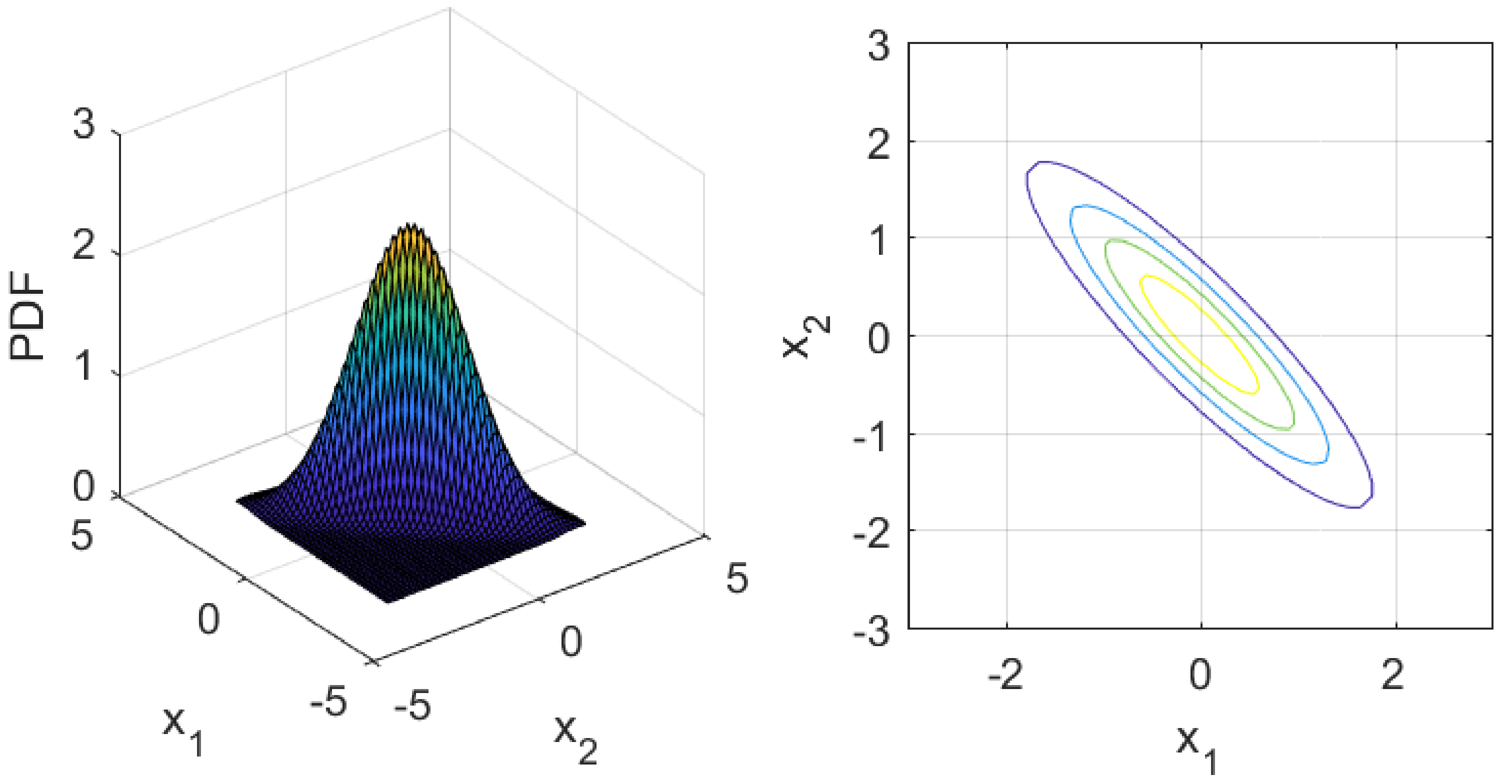

Figure 10.15: Left: Bivariate standard Gaussian distribution. Right: Contour Level. Parameter $\rho = -0.9$.

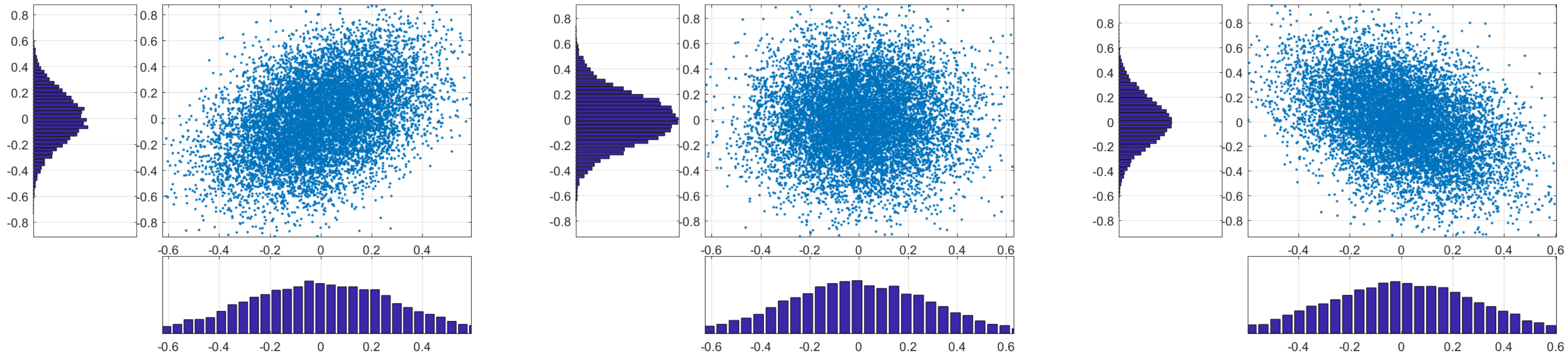

Figure 10.16: Jointly simulated Gaussian random variables (both having zero mean and unit standard deviation) with different correlation. Left: $\rho = 0.9$. Mid: $\rho = 0$. Right: $\rho = -0.9$.

An important fact about sum of Gaussian random variables

Fact 85 *The marginal distributions of a vector* **X** *can all be Gaussian without the joint being multivariate Gaussian.*

Example 86 *Let* $X_1 \sim \mathcal{N}(0,1)$ *and*

$$X_2 = \begin{cases} X_1 & if \quad -c < X_1 < c \\ \\ -X_1 & elsewhere \end{cases}$$

We have

$$\Pr(X_2 < x) = \begin{cases} \Pr(X_1 < x) & if \quad -c < X_1 < c \\ \\ \Pr(X_1 > -x) & elsewhere \end{cases}$$

and the density is

$$f_{X_2}(x) = \begin{cases} f_{X_1}(x) & if \quad -c < X_1 < c \\ \\ f_{X_1}(x) & elsewhere \end{cases} = f_{X_1}(x)$$

and therefore it is Gaussian as well. This is illustrated in Figure (10.17).

Fact 87 *However, the sum of two Gaussian random variables is in general not Gaussian.*

Example 88 *Let us consider the previous example and let us build the new random variable*

$$Y = X_1 + X_2 = \begin{cases} 2X_1 & if \quad -c < X_1 < c \\ \\ 0 & elsewhere \end{cases}$$

that clearly it is not Gaussian, as shown in Figure (10.18).

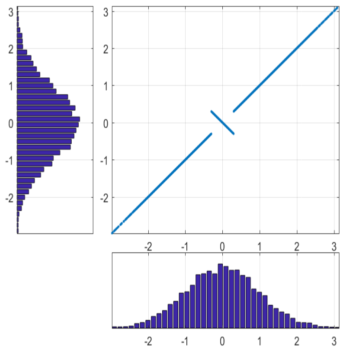

Figure 10.17: Margins are Gaussian, the joint distribution is not Gaussian. We represent the marginal distribution of X_1 and X_2 and their joint distribution as in example 86.

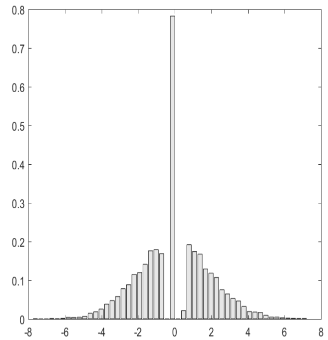

Figure 10.18: The distribution of the sum of $Y = X_1 + X_2$, where both random variables are Gaussian, but they are not jointly Gaussian as discussed in example 88. The random variable sum Y is not Gaussian.

10.9. Simulating Random Variables

Many applications entail sampling random variables from pre-specified distributions. A very popular general technique to achieve this is the inverse transform method. This method is implementable if the cumulative density function and its inverse can be computed without difficulties. The idea is formalized in Fact 89 and illustrated in Figure (10.19).

Fact 89 *Suppose we need to simulate a random variable X with cumulative distribution function F_X, i.e. such that $\mathbb{P}(X \leq x) = F_X(x)$. Then, the inverse transform method sets*

$$X = F_X^{-1}(U), \quad U \sim Unif[0,1],$$

where F_X^{-1} is the inverse of F_X and $Unif[0,1]$ denotes the uniform distribution on $[0,1]$.

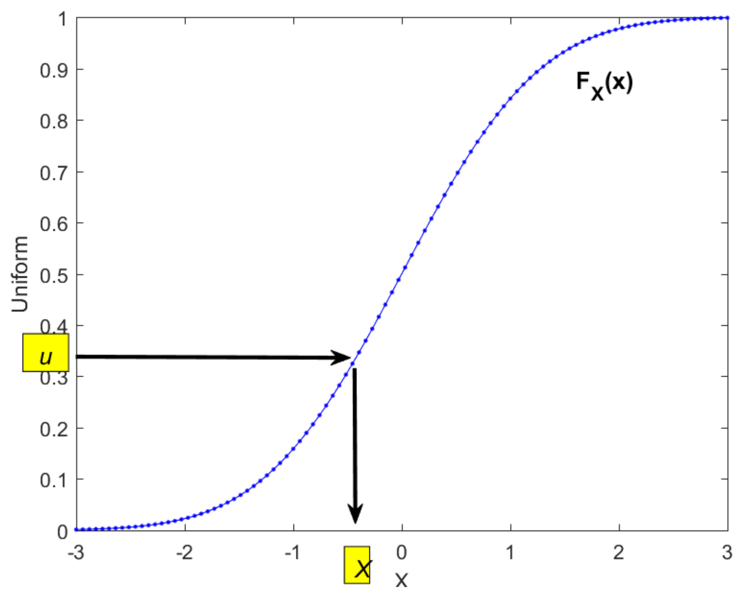

Figure 10.19: Inverse transform method for a hypothetical CDF F_X.

10.9.1. Example: Sampling from the Normal Distribution

We refer to the properties presented in section 10.1.

- In Excel simulation of Gaussian r.v.'s can be performed by inputting in a cell the command =NORMSINV(RAND())
- In Matlab, we can simulate Gaussian random numbers by simulating Uniform random numbers through the command

```
>U=rand(1,1)
```

  and then applying to it the inverse Gaussian CDF

```
>Z=norminv(U,0,1);
```

- If we are interested in generating Gaussian random variables with assigned mean μ and standard deviation σ we can use

```
>X=norminv(U,mu,sigma);
```

  This is illustrated in detail below at page 253.

- A more direct approach in Matlab is through the command

```
>Z=randn(1,1);
```

- If we are interested in generating Gaussian random variables with assigned mean μ and standard deviation σ we just do

```
>X=mu+Z*sigma;
```

  This is illustrated in detail below at page 255.

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% SIMULATION OF STD GAUSSIAN RANDOM VARIABLES BY INVERSION %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all; close all
%assign parameters
NSim=10^6;
%simulate a sample of Uniform(0,1) random variables
R=rand(1,NSim);
%simulate the sample of standard Gaussian r.v. by inversion
X = norminv(R,0,1);
%make the plot
h=figure('Color', [ 1 1 1]);
subplot(4,4,[1 5 9]);
u=linspace(0,1,50);
hist(R,u); view(90,-90)
title('Uniform r.v.')
set(gca,'XTick',[0:0.1:1])
set(gca,'YTickLabel',''); grid on
subplot(4,4,[2:4 6:8 10:12]);
x=linspace(-4,4,100);
f = kSDEnsity(X,x,'function','cdf');
plot(x,f);
title('Inversion of the cdf')
set(gca,'XTick',[-4:1:4])
set(gca,'YTick',[0:0.1:1]); grid on
subplot(4,4,[14:16]);
histfit(X,100)
title('Standard Gaussian r.v.')
set(gca,'XTick',[-4:1:4])
set(gca,'YTickLabel',''); grid on
```

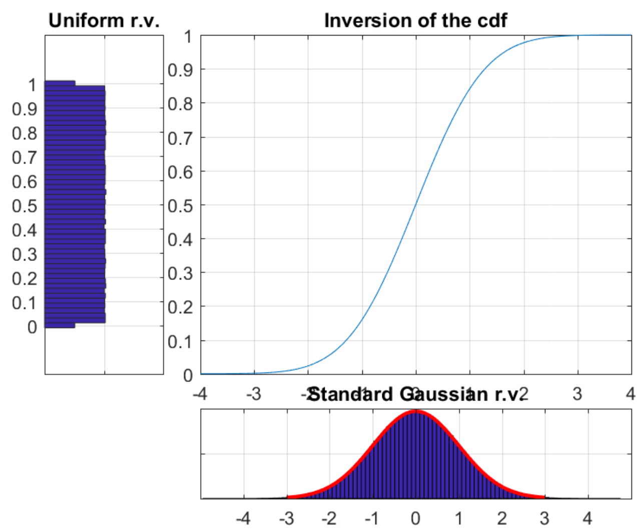

Figure 10.20: Sampling from the standard Gaussian distribution.

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% SIMULATION OF GAUSSIAN RANDOM VARIABLE BY INVERSION %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all; close all
%assign parameters
NSim=10^6; mu=0.2; sg=0.1;
%simulate a sample of Uniform(0,1) random variables
R=rand(1,NSim);
%simulate the sample of non standard Gaussian r.v. by inversion
X = norminv(R,mu,sg);
h=figure('Color', [ 1 1 1]);
subplot(4,4,[1 5 9]);
u=linspace(0,1,50);
hist(R,u); view(90,-90)
title('Uniform r.v.')
set(gca,'XTick',[0:0.1:1])
set(gca,'YTickLabel',''); grid on
subplot(4,4,[2:4 6:8 10:12]);
x=linspace(-0.3,0.5,100);
f = kSDEnsity(X,x,'function','cdf');
plot(x,f);
title('Inversion of the cdf')
set(gca,'XTick',[-0.3:0.1:0.5])
set(gca,'YTick',[0:0.1:1]); grid on
subplot(4,4,[14:16]);
histfit(X,100)
title('Gaussian r.v.'); xlim([-0.3 0.5]);
set(gca,'XTick',[-0.3:0.1:0.5])
set(gca,'YTickLabel',''); grid on
```

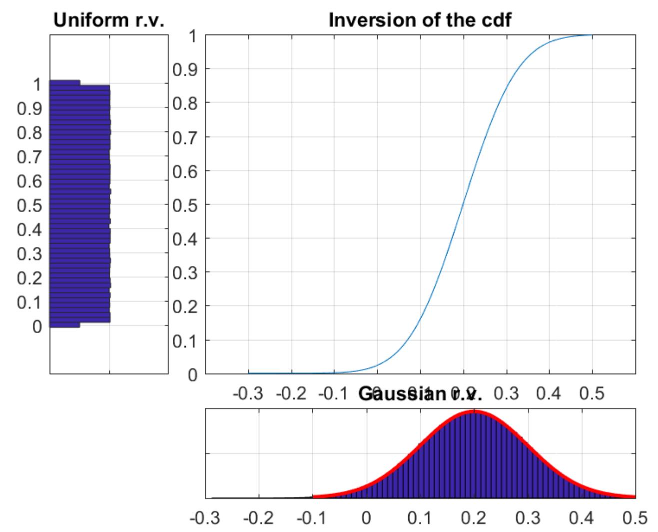

Figure 10.21: Sampling from the Gaussian distribution with $\mu = 0.2$ and $\sigma = 0.1$.

10.9.2. Example: Sampling from the Lognormal Distribution

We can exploit the relationship between Gaussian and Lognormal distribution. In the following lines we simulate 10000 Lognormal variables with parameters $\mu = 0.05$ and $\sigma = 0.2$.

```
>mu=0.05;
>sigma=0.2
>X=mu+Z*randn(10000,1);
>Y=exp(X);
```

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% SIMULATION OF LOGNORMAL RANDOM VARIABLES %%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all;
close all
%assign parameters
NSim=10^6;
mu=0.05;
sigma=0.2;
%simulate the sample of Gaussian r.v.
X = mu+sigma*randn(NSim,1);
%simulate the Lognormal
%by exponential transformation
Y=exp(X);
h=figure('Color', [ 1 1 1]);
histfit(Y,100,'lognormal');
title('Simulated Lognormal Random Variables')
```

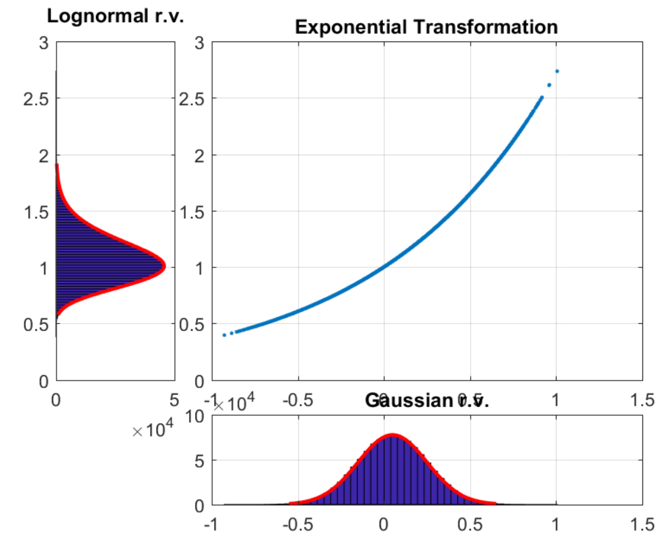

Figure 10.22: Sampling from the Lognormal distribution with parameters $\mu = 0.05$ and $\sigma = 0.2$.

10.9.3. Example: Sampling from the Poisson Distribution

We refer to the properties presented in section 10.5. This is illustrated in detail below at page 257.

10.9.4. Example: Sampling from the Gamma Distribution

We refer to the properties presented in section 10.7. This is illustrated in detail below at page 258.

10.9.5. Example: Sampling from the Chi-Square Distribution

A Chi-Square distribution is a special case of the Gamma distribution with shape parameter α. Therefore, the simulation turns out to be a special case of the Gamma. However, notice that Matlab has a built-in function to sample directly from the chi-square distribution In the following lines we simulate 10000 chi-square random variables with degrees of freedom 5.

```
>df=5;
>chi2=chi2rnd(df,10000,1);
```

10.9.6. Example: Sampling from the Non-Central Chi-Square Distribution

Matlab provides a routine that allows to compute the inverse CDF of this distribution, (ncx2inv(p,df,nc) where p is the probability level, df is the number of degrees of freedom and nc is the parameter of non-centrality). Matlab also provides a random number generator for this distribution. The function ncx2rnd(df,nc, m,n) returns a matrix mxn of random numbers chosen from the non-central chi-square distribution with degrees of freedom df and positive noncentrality parameters nc. In the following lines we simulate 10000 chi-square random variables with degrees of freedom 5.9 and parameter of non-centrality equal to 4.5.

```
>df=5.9;
>nc=4.5
>chi2nc=ncx2rnd(df,nc, 10000,1);
```

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% SIMULATION OF POISSON RANDOM VARIABLE BY INVERSION  %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all; close all
%assign parameters
NSim=10^6; lambdaP=10;
%simulate a sample of Uniform(0,1) random variables
R=rand(1,NSim);
%simulate the sample of Poisson r.v. by inversion
X = poissinv(R,lambdaP);
h=figure('Color', [ 1 1 1]);
subplot(4,4,[1 5 9]);
u=linspace(0,1,50);
hist(R,u); view(90,-90)
title('Uniform r.v.')
set(gca,'XTick',[0:0.1:1])
set(gca,'YTickLabel','')
grid on
subplot(4,4,[2:4 6:8 10:12]);
x=(0:1:30);
f = kSDEnsity(X,x,'function','cdf');
plot(x,f,'-o');
title('Inversion of the cdf')
set(gca,'XTick',[0:5:30])
set(gca,'YTick',[0:0.1:1]); grid on
subplot(4,4,[14:16]);
histfit(X,15,'poisson')
title('Poisson r.v.')
set(gca,'XTick',[0:5:30])
set(gca,'YTickLabel',''); grid on
```

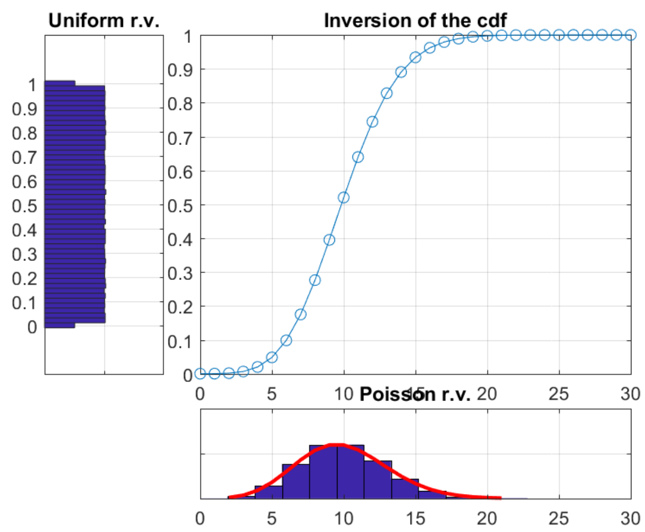

Figure 10.23: Sampling from the Poisson distribution with $\lambda = 10$.

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% SIMULATION OF GAMMA RANDOM VARIABLE BY INVERSION %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all; close all
%assign parameters
NSim=10^6; alpha=2; lambdaG=0.5;
%simulate the corresponding sample of Gamma r.v. by inversion
X = gaminv(R,alpha,lambdaG^(-1));
h=figure('Color', [ 1 1 1]); subplot(4,4,[1 5 9]);
u=linspace(0,1,50); hist(R,u); view(90,-90);
title('Uniform r.v.')
set(gca,'XTick',[0:0.1:1])
set(gca,'YTickLabel',''); grid on
subplot(4,4,[2:4 6:8 10:12]);
x=linspace(0,20,200);
f = kSDEnsity(X,x,'function','cdf');
plot(x,f);
title('Inversion of the cdf')
set(gca,'XTick',[0:2:20])
set(gca,'YTick',[0:0.1:1]); grid on
subplot(4,4,[14:16]); histfit(X,100,'gamma')
title('Gamma r.v.'); xlim([0 20]);
set(gca,'XTick',[0:2:20])
set(gca,'YTickLabel',''); grid on
```

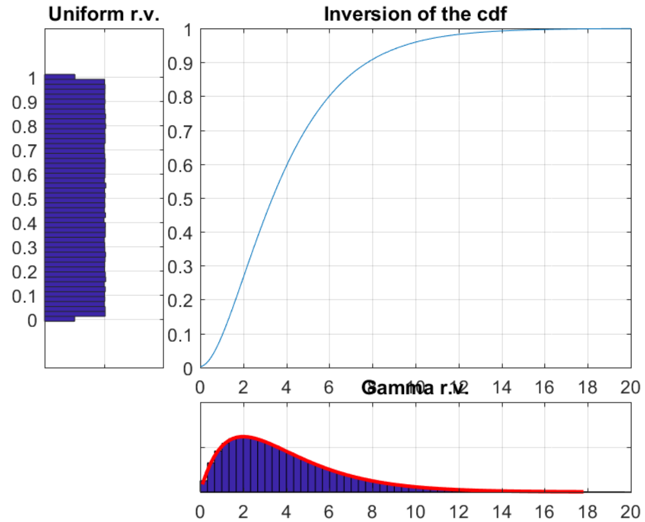

Figure 10.24: Sampling from the Gamma distribution with $\alpha = 2, \lambda = 0.5$.

10.9.7. Example: Sampling from the Multivariate Normal Distribution

We refer to the properties presented in section 10.8. Further, we note the following.

Fact 90 $\mathbf{X} \sim \mathcal{N}\,(\boldsymbol{\mu},\boldsymbol{\Sigma})$ *if and only if it admits the representation*

$$\mathbf{X} = \boldsymbol{\mu} + \mathbf{AZ}, \tag{10.1}$$

where $\mathbf{Z} = (Z_1, ..., Z_N)$ *is a a multivariate standard normal distribution and* $\mathbf{A}$ *is a* $N \times N$ *lower triangular matrix with strictly positive diagonal entries.*

Fact 91 *It follows that:*

$$\mathbf{\Sigma} = \mathbb{V}ar\left(\mathbf{X}\right) = \mathbb{V}ar\left(\mathbf{AZ}\right) = \mathbf{A}\mathbb{V}ar(\mathbf{Z})\,\mathbf{A}^{'} = \mathbf{A}\mathbb{I}\mathbf{A}^{'} = \mathbf{AA}',$$

and we have the so called Cholesky decomposition of $\mathbf{\Sigma}$

$$\mathbf{\Sigma} = \mathbf{AA}'$$

and this guarantees that $\mathbf{\Sigma}$ *is semidefinite positive. If rank* $(\mathbf{A}) = N$, $\mathbf{\Sigma}$ *is definite positive. This is known as Cholesky decomposition*

Example 92 *Let*

$$\mathbf{\Sigma} = \begin{bmatrix} 0.04 & 0.024 \\ 0.024 & 0.09 \end{bmatrix}.$$

In order to find its Cholesky decomposition, we need to look for a 2x2 lower triangular matrix $\mathbf{A}$ *such that*

$$\begin{bmatrix} 0.04 & 0.024 \\ 0.024 & 0.09 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{21} \\ 0 & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}^2 & a_{11}a_{21} \\ a_{11}a_{21} & a_{21}^2 + a_{22}^2 \end{bmatrix},$$

i.e. we have to set

$$\left\{ \begin{array}{c} a_{11}^2 = 0.04 \\ a_{11}a_{21} = 0.024 \\ a_{21}^2 + a_{22}^2 = 0.09 \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} a_{11} = \sqrt{0.04} = 0.2 \\ a_{21} = \frac{0.024}{0.2} = 0.12 \\ a_{22} = \sqrt{0.09 - (0.12)^2} = \sqrt{0.0756} = 0.27495 \end{array} \right.$$

and therefore

$$\mathbf{A} = \begin{bmatrix} 0.2 & 0 \\ 0.12 & 0.2749545 \end{bmatrix}.$$

- The Cholesky decomposition is at the basis of simulation methods for multivariate Gaussian random variables.

- It can be verified that if
$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix},$$
then
$$\mathbf{A} = \begin{bmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{bmatrix}.$$

- This means that in order to perform Monte Carlo simulation we need to simulate Z_1 and Z_2 independently and according to a standard normal random variable and then to set
$$X_1 = \mu_1 + \sigma_1 Z_1,$$
$$X_2 = \mu_2 + \rho\sigma_2 Z_1 + \sigma_2\sqrt{1-\rho^2}Z_2.$$

Example 93 *In the bivariate case, we can apply the Cholesky decomposition of previous example and obtain*
$$\mathbf{\Sigma} = \begin{bmatrix} 0.04 & 0.024 \\ 0.024 & 0.09 \end{bmatrix} \Longrightarrow \mathbf{A} = \begin{bmatrix} 0.2 & 0 \\ 0.4 \times 0.3 & 0.3 \times \sqrt{1-0.4^2} \end{bmatrix}$$
so that correlated Gaussian r.v's are simulated according to
$$X_1 = \mu_1 + 0.2 \times Z_1,$$
$$X_2 = \mu_2 + 0.4 \times 0.3 \times Z_1 + 0.3 \times \sqrt{1-0.4^2} \times Z_2.$$

In Matlab we can simulate a multivariate Gaussian distribution using the command mvnrnd that takes as parameters the mean vector, the covariance matrix and the number of simulations. This is illustrated in details in the following Matlab script.

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%% SIMULATION OF BIVARIATE NORMAL RANDOM VARIABLE %%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all
close all
%assign parameters
NSim=10^6;
rho=0.9; mu1=0; mu2=0;
s1=0.3^2; s2=0.5^2; s12=rho*(s1*s2)^0.5;
%simulate the corresponding sample
%of bivariate Gaussian r.v.
Mu=[mu1 mu2]; VC=[s1 s12; s12 s2];
X = mvnrnd(Mu,VC,10000);
X1=X(:,1); X2=X(:,2);
h=figure('Color', [ 1 1 1]);
subplot(4,4,[1 5 9]);
histfit(X1,100)
set(gca,'YTickLabel','')
view(90,-90)
grid on
subplot(4,4,[2:4 6:8 10:12]);
plot(X2,X1,'+')
grid on
subplot(4,4,[14:16]);
histfit(X2,100)
set(gca,'YTickLabel','')
grid on
```

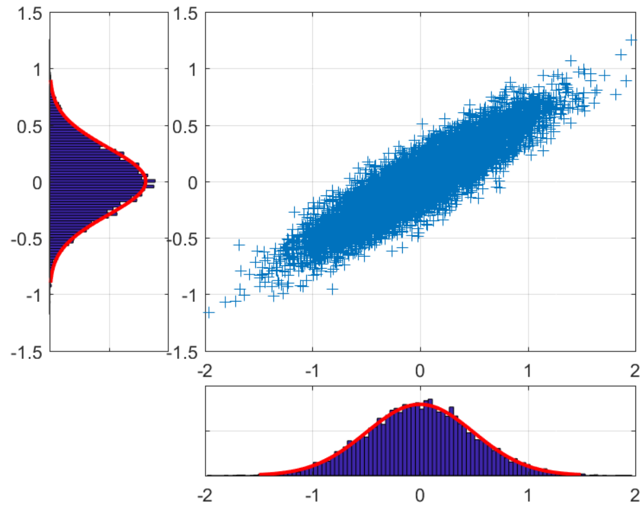

Figure 10.25: Sampling from the bivariate Gaussian distribution (positive correlation).